J. Fluid Mech. (1967), *vol.* 29, *part* 4, *pp.* 731–743
Printed in Great Britain

Peristaltic motion

By J. C. BURNS
Department of Applied Mathematics, Australian National University
AND T. PARKES
Department of Mathematics, Royal Military College, Duntroon

(Received 22 November 1966)

The flow of a viscous fluid through axially symmetric pipes and symmetrical channels is investigated under the assumption that the Reynolds number is small enough for the Stokes flow approximations to be made. It is assumed that the cross-section of the pipe or channel varies sinusoidally along the length. The flow is produced by a prescribed pressure gradient and by the variation in cross-section that occurs during the passage of a prescribed sinusoidal peristaltic wave along the walls. The theory is applied in particular to two extreme cases, peristaltic motion with no pressure gradient and flow under pressure along a pipe or channel with fixed walls and sinusoidally varying cross-section. Perturbation solutions are found for the stream function in powers of the ratio of the amplitude of the variation in the pipe radius or channel breadth to the mean radius or breadth respectively. These solutions are used to calculate, in particular, the flux through the pipe or channel for a given wave velocity in the first case and for a given pressure gradient in the second case. With a suitable notation it is possible to combine the analysis required for the two cases of pipe and channel flow.

1. Introduction

The study of the flow of an incompressible viscous liquid is greatly simplified if discussion is limited to Stokes flow in which the Reynolds number is small enough for the inertia forces to be neglected in comparison with the viscous forces so that the equations of motion become linear.

In this paper we consider two-dimensional flow through a symmetrical channel and axially symmetric flow through a pipe of circular cross-section. In each case the boundary varies sinusoidally.

Two causes of motion are considered. It will be assumed that there is a prescribed pressure gradient along the pipe or channel and that a progressive wave passes along the walls. It will be seen that, provided the frequency of oscillation is small enough, this peristaltic motion is governed by the usual equations for steady Stokes flow. Thus the two extreme cases, of motion caused solely by the variation in cross-section and of motion under a pressure gradient when the walls are fixed can be treated together. Moreover, the two cases of pipe flow and channel flow can be treated together by taking advantage of the notation of generalized axi-symmetric potential theory to develop the theory in a form which is applicable to each case, leaving only the detailed calculations to be carried out separately.

It will be convenient, where it is not necessary to distinguish between channel flow and pipe flow, to use 'tube' to denote either the symmetric channel or the axisymmetric pipe and 'radius of the tube' to denote half the breadth of the channel or the radius of the pipe.

The problem is solved by expanding the stream function which determines the flow as a Fourier series involving two infinite sets of unknown coefficients. The boundary conditions on the wall of the tube give a set of linear equations which can be solved for these coefficients. Following closely the method used by Taylor (1951) in a similar problem, we obtain a perturbation solution in which these coefficients are obtained as power series in η/h, the ratio of the amplitude of the variation of the tube radius to the mean tube radius.

Although peristaltic motion of a viscous fluid through pipes and channels does not seem to have been discussed previously, the particular case of flow through a fixed tube under a prescribed pressure gradient has been treated by a number of authors. Langlois (1964) has discussed flow along channels of varying breadth and obtained approximate solutions in several different cases. Gheorgita (1959) has found solutions to the first-order in η for symmetrical channels in which the breadth varies along the length according to a cosine law and has also given first-order solutions in cases in which the distance of each wall from the centre plane varies periodically along the length with the same frequency but the channel is not symmetrical. Belinfante (1962) has considered flow of a viscous fluid along pipes and channels in which the radius or breadth varies along the length according to a cosine law. He also has obtained solutions for the Stokes flow approximation correct to the first-order in η/h. He used these solutions as a basis for solutions of the Navier–Stokes equations in powers of the Reynolds number of the flow. He remarks that he has also obtained solutions of the Stokes flow problem to the second-order in η/h for both pipe and channel flow. Burns (1965) has used the methods of the present paper to obtain results for both pipes and channels, the essential features of which are included (with some corrections) as a special case of the general problem of peristaltic flow with a pressure gradient considered here.

2. Statement of the problem

The wall of the tube is defined by the equation

$$y = h + \eta \cos l(x - \sigma t) \tag{1}$$

so that a progressive wave of amplitude η, velocity σ and wavelength $\lambda = 2\pi/l$ passes along the tube in the positive x-direction. It will frequently be convenient to write $z = x - \sigma t$. The (x, y)-plane is a meridian plane of the tube, the axis OX being along the axis of symmetry and the axis OY normal to OX. Let the velocity components in the x, y directions be u, v respectively (see figure 1). If $\sigma = 0$, the wall of the tube is a fixed cosine wave.

Steady Stokes flow in the absence of body forces is determined by the equations

$$\nabla . \mathbf{v} = 0, \quad \mu \nabla \times \boldsymbol{\omega} = -\nabla p \tag{2}$$

in which $\mathbf{v}$ is the velocity, μ is the viscosity, p is the pressure and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity. These same equations are valid also for the unsteady flow, provided the oscillations of the walls are such that $\sigma/\nu l = o(1)$ where ν is the kinematic viscosity (see Rosenhead 1963, p. 169, for example.)

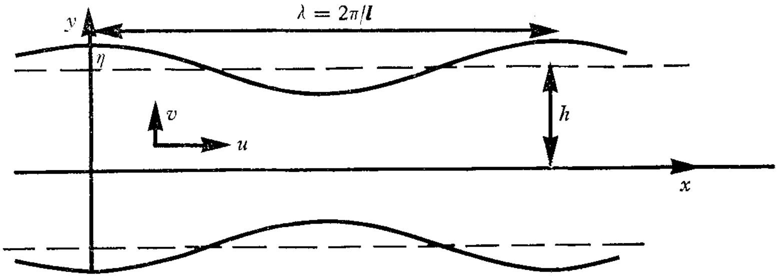

FIGURE 1.

If we introduce a stream function ψ and let $\omega = |\boldsymbol{\omega}|$, the notation introduced by Weinstein (1953) enables us to write the equations satisfied by ψ and ω as

$$L_{-k}(\psi) = -y^k\omega, \tag{3}$$

$$L_{-k}(y^k\omega) = 0, \tag{4}$$

where $$L_{-k} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{k}{y}\frac{\partial}{\partial y}$$

and $k = 0, 1$ according as the flow is two-dimensional or axi-symmetrical.

The following conditions must be satisfied. On the axis of symmetry we can take $\psi = 0$ and we must have $\omega \to 0$ as $y \to 0$. On the outer boundary of the flow the fluid must have the same velocity as the wall of the tube. It will be assumed in the first instance that the particles of the tube wall move in straight lines perpendicular to the axis of the tube so that the boundary condition is

$$y^{-k}\frac{\partial \psi}{\partial y} = u = 0,$$

$$-y^{-k}\frac{\partial \psi}{\partial z} = v = \frac{\partial y}{\partial t} = l\sigma\eta \sin lz \quad \text{on} \quad y = h+\eta\cos lz. \tag{5}$$

This condition requires that the wall of the tube be extensible. A modified boundary condition will be considered later.

The problem is to solve (3) and (4) for ψ and ω subject to these conditions on the axis of symmetry and the wall of the tube.

Since the boundary varies periodically with z and is symmetrical about $z = 0$, it follows that both ψ and ω are even periodic functions of z with wavelength λ. It is easily shown that the pressure is constant on the sections $z = 0, \lambda, 2\lambda \ldots$ and that the pressure difference between successive points of maximum cross-section is always the same. The pressure gradient which produces the flow must be assumed known and we shall let the pressure drop over a wavelength have the prescribed value P.

3. Fourier series for the stream function

The functions $\psi(z,y)$ and $\omega(z,y)$, being even, periodic functions of z of wave-length $\lambda = 2\pi/l$, can be expressed as Fourier cosine series in the form

$$\psi(z,y) = \sum_{n=0}^{\infty} \psi_n(y) \cos nlz, \tag{6}$$

$$\omega(z,y) = \sum_{n=0}^{\infty} \omega_n(y) \cos nlz. \tag{7}$$

If (6) and (7) are substituted in (3) and (4) and the coefficients of terms in $\cos nlz$ compared, then for $n \geqslant 0$, $\psi_n(y)$ and $\omega_n(y)$ satisfy the equations

$$\frac{d^2\psi_n}{dy^2} - \frac{k}{y}\frac{d\psi_n}{dy} - n^2l^2\psi_n = -y^k\omega_n, \tag{8}$$

$$\frac{d^2(y^k\omega_n)}{dy^2} - \frac{k}{y}\frac{d(y^k\omega_n)}{dy} - n^2l^2(y^k\omega_n) = 0. \tag{9}$$

These equations have to be solved under the conditions that $\psi_n \to 0$ and $\omega_n \to 0$ as $y \to 0$.

The solutions to (9) satisfying these conditions are

$$y^k\omega_0 = -A_0 y^{k+1}$$

and for $n \geqslant 1$

$$y^k\omega_n = -A_n y^{\frac{1}{2}(k+1)} I_{\frac{1}{2}(k+1)}(nly), \tag{10}$$

where A_0, A_n are arbitrary constants and $I_\nu(x)$ is a modified Bessel function of the first kind of order ν.

When ω_n in (8) is replaced by the expressions given in (10) then the resulting equations are of a type for which particular integrals can be found (Burns 1966) and the complementary functions are of course the general solutions of (9). The functions $\psi_n(y)$ which satisfy these equations and the condition $\psi_n \to 0$ as $y \to 0$ are found without difficulty and when these are substituted in (6) the resulting expression for $\psi(z,y)$ is

$$\psi(z,y) = \frac{A_0}{2(k+3)} y^{k+3} + B_0 y^{k+1} + \sum_{n=1}^{\infty} y^{\frac{1}{2}(k+1)} \left[\frac{A_n}{2nl} y\, I_{\frac{1}{2}(k-1)}(nly) + B_n I_{\frac{1}{2}(k+1)}(nly)\right] \cos nlz. \tag{11}$$

The arbitrary constants A_n, B_n for $n \geqslant 0$ must be determined from the condition (5), that the fluid on the boundary has the same velocity as the wall of the tube, together with the condition that the pressure drop per wavelength is P.

Since the pressure is constant over the sections $z = 0, \lambda$ it follows that the pressure drop between these sections can be obtained by integrating $\partial p/\partial z$ along the line $y = 0$.

Equation (2) gives

$$\frac{\partial p}{\partial z} = -\mu y^{-k} \frac{\partial}{\partial y}(y^k\omega)$$

so that using (7) and (10) we find that

$$\left(\frac{\partial p}{\partial z}\right)_{y=0} = \mu(k+1)A_0 + \sum_{n=1}^{\infty} f_n \cos nlz, \tag{12}$$

where the coefficients f_n are constants. Integration of (12) from $z = 0$ to $z = 2\pi/l$ gives

$$A_0 = -\frac{Pl}{2\mu\pi(k+1)}. \tag{13}$$

Thus the constant A_0 is known in terms of the prescribed pressure gradient. The remaining boundary condition leads to equations sufficient to determine the constants B_0, A_n, B_n $(n \geqslant 1)$ in terms of A_0 and σ.

At this point it becomes convenient to give separate (although closely similar) discussions of the two cases of channel flow and pipe flow.

For channel flow, $k = 0$ and the stream function becomes

$$\psi(z,y) = \frac{A_0}{6}y^3 + B_0 y + \left(\frac{2}{\pi l}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{2}} \left\{\frac{A_n}{2nl} y \cosh nly + B_n \sinh nly\right\} \cos nlz. \tag{14}$$

The replacement of the Bessel functions $I_{\frac{1}{2}}(nly)$ and $I_{-\frac{1}{2}}(nly)$ by expressions involving $\cosh nly$ and $\sinh nly$ leads to a considerable simplification in the detailed analysis which follows.

For pipe flow, $k = 1$ and the stream function becomes

$$\psi(z,y) = \frac{A_0}{8}y^4 + B_0 y^2 + \sum_{n=1}^{\infty} y \left\{\frac{A_n}{2nl} y I_0(nly) + B_n I_1(nly)\right\} \cos nlz. \tag{15}$$

It is convenient to introduce new coefficients as follows.

For channel flow, let

$$a_0 = \tfrac{1}{2}A_0, \quad b_0 = B_0$$

and for $n \geqslant 1$,
$$a_n = \left(\frac{2}{n\pi l}\right)^{\frac{1}{2}} \frac{A_n}{2nl}, \quad b_n = \left(\frac{2}{n\pi l}\right)^{\frac{1}{2}} B_n; \tag{16}$$

and for pipe flow, let
$$a_0 = \tfrac{1}{2}A_0, \quad b_0 = 2B_0, \tag{17}$$

and, for $n \geqslant 1$,
$$a_n = \frac{A_n}{2nl}, \quad b_n = B_n.$$

The stream function for channel flow then becomes

$$\psi(z,y) = \tfrac{1}{3}a_0 y^3 + b_0 y + \sum_{n=1}^{\infty} \{a_n y \cosh nly + b_n \sinh nly\} \cos nlz, \tag{18}$$

where b_0; a_n, b_n $(n \geqslant 1)$ are to be found in terms of a_0 and σ from the conditions

$$\partial\psi/\partial y = 0 \quad \text{and} \quad \partial\psi/\partial z = -l\sigma\eta \sin lz \quad \text{on} \quad y = h + \eta \cos lz, \tag{19}$$

which are obtained from (5) by putting $k = 0$.

The stream function for pipe flow becomes

$$\psi(z,y) = \tfrac{1}{4}a_0 y^4 + \tfrac{1}{2}b_0 y^2 + \sum_{n=1}^{\infty} y\{a_n y I_0(nly) + b_n I_1(nly)\}\cos nlz, \tag{20}$$

where b_0; a_n, b_n ($n \geqslant 1$) are to be found in terms of a_0 and σ from the conditions

$$\frac{1}{y}\frac{\partial\psi}{\partial y} = 0 \quad \text{and} \quad \frac{1}{y}\frac{\partial\psi}{\partial z} = -l\sigma\eta\sin lz \quad \text{on} \quad y = h+\eta\cos lz, \tag{21}$$

which are obtained from (5) by putting $k = 1$.

4. Evaluation of the coefficients a_n, b_n

The boundary conditions (19), (21) at the tube wall, which in each case has equation $y = h+\eta\cos lz = y_1$ say, lead to the following equations.

For channel flow,

$$a_0 y_1^2 + b_0 + \sum_{n=1}^{\infty}[(a_n + nlb_n)\cosh nly_1 + nla_n y_1 \sinh nly_1]\cos nlz = 0 \tag{22}$$

and

$$\sum_{n=1}^{\infty}[a_n y_1 \cosh nly_1 + b_n \sinh nly_1]\, n\sin nlz = \eta\sigma\sin lz; \tag{23}$$

for pipe flow,

$$a_0 y_1^2 + b_0 + \sum_{n=1}^{\infty}[(2a_n + nlb_n) I_0(nly_1) + nla_n y_1 I_1(nly_1)]\cos nlz = 0 \tag{24}$$

and

$$\sum_{n=1}^{\infty}[a_n y_1 I_0(nly_1) + b_n I_1(nly_1)]\, n\sin nlz = \eta\sigma\sin lz. \tag{25}$$

For channel flow, $\cosh nly_1$ and $\sinh nly_1$ and for pipe flow, $I_0(nly_1)$ and $I_1(nly_1)$ are expanded in powers of $\cos lz$. Substitution in (22), (23) and (24), (25) leads, in each case, to terms of the form $\cos^p lz \cos nlz$ and $\cos^p lz \sin nlz$ which are expanded in Fourier cosine and sine series respectively. Finally, the coefficients of terms in $\cos rlz$ and $\sin rlz$ in the resulting equations are equated and linear equations for b_0; a_n, b_n ($n \geqslant 1$) are obtained. In each case, these are of the form

$$\sum_{n=1}^{\infty} p_{nr} a_n + \sum_{n=1}^{\infty} q_{nr} b_n = 0 \quad (r = 1,2,3,\ldots),$$

$$b_0 + \sum_{n=1}^{\infty} h_{n0} a_n + \sum_{n=1}^{\infty} k_{n0} b_n = c_0,$$

$$\sum_{n=1}^{\infty} h_{nr} a_n + \sum_{n=1}^{\infty} k_{nr} b_n = c_r \quad (r = 1,2,3,\ldots), \tag{26}$$

where all the coefficients p_{nr}, q_{nr}, h_{nr}, k_{nr}, c_r are known. For the purposes of the perturbation solution used here, these are expanded in powers of $l\eta$ and the leading term of the series for each of the first four involves $l\eta$ raised to the power $|n-r|$ while c_r is of the order $(l\eta)^r$. It follows that a_n and b_n are of order $(l\eta)^n$ and it turns out that they are obtained as series in the form

$$\left.\begin{aligned} a_n &= \sum_{t=0}^{\infty} \alpha_{n,n+2t}(l\eta)^{n+2t} \quad (n \geqslant 1);\\ b_n &= \sum_{t=0}^{\infty} \beta_{n,n+2t}(l\eta)^{n+2t} \quad (n \geqslant 0).\end{aligned}\right\} \tag{27}$$

At this stage we assume that $l\eta$ is small and calculate $\psi(z,y)$ to order $(l\eta)^n$. Imposing this restriction, and comparing coefficients of powers of $l\eta$ in (26), gives a set of equations which can be solved for α_{nr} and β_{nr}.

To find $\psi(z,y)$ to order $(l\eta)^n$, $\frac{1}{2}(n+1)(n+2)$ equations are needed but these can be solved in pairs. Thus to find ψ to order $(l\eta)^4$ needs 15 equations. This is the order of most of the calculations in the rest of the paper.

5. Calculation of the flux through the tube

To find the flux through the tube it is necessary to evaluate the stream function $\psi(z,y)$ at a point on the boundary $y = h+\eta \cos lz$. For any value of x, this flux varies periodically with the time. In fact because a_n, b_n are determined as power series in $l\eta$ it follows that the flux ψ is also a power series in $l\eta$. Alternatively we can write ψ as a power series in η/h, i.e.

$$\psi = \psi_0+\psi_1(\eta/h)+\psi_2(\eta/h)^2+\ldots,$$

where ψ_n is a periodic function of z.

What is wanted is the average flux per cycle and this can be found by integrating ψ at a point on the boundary over one period. Doing this removes all the odd powers of η/h for it is easily seen that

$$\int_0^\lambda \psi_1 dz = \int_0^\lambda \psi_3 dz = \ldots = 0.$$

The mean flux is then

$$\overline{\psi} = \overline{\psi}_0+\overline{\psi}_2(\eta/h)^2+\overline{\psi}_4(\eta/h)^4+\ldots \tag{28}$$

and the expressions obtained for $\overline{\psi}_0, \overline{\psi}_2, \overline{\psi}_4$ are as follows:

(*a*) for channel flow,

$$\overline{\psi}_0 = -\tfrac{2}{3}a_0 h^3, \quad \overline{\psi}_2 = -h^3[a_0+l^2\alpha_{11}\sinh lh+\tfrac{1}{2}l^2\sigma],$$

$$\overline{\psi}_4 = -l^3h^4[\tfrac{1}{8}\alpha_{11}(lh\sinh lh+2\cosh lh)+\alpha_{13}\,lh\sinh lh + \tfrac{1}{2}\alpha_{22}(2lh\cosh 2lh+\sinh 2lh)]; \tag{29}$$

(*b*) for pipe flow

$$\overline{\psi}_0 = -\tfrac{1}{4}a_0h^4, \quad \overline{\psi}_2 = -\tfrac{1}{4}h^4[3a_0+2l^2\alpha_{11}I_1(lh)+l^2\sigma],$$

$$\overline{\psi}_4 = -h^4[\tfrac{3}{32}a_0+\tfrac{1}{16}l^3h\alpha_{11}(3I_0(lh)+lhI_1(lh))+\tfrac{1}{2}l^4h^2\alpha_{13}I_1(lh) + \tfrac{1}{4}l^3h\alpha_{22}(2lhI_0(2lh)+I_1(2lh))+\tfrac{1}{16}l^2\sigma]. \tag{30}$$

In each case α_{11}, α_{13}, α_{22}, are obtained as indicated in § 4.

6. Numerical calculations of flux, streamlines and velocity distribution

In the numerical calculations the two causes of motion have been treated separately, i.e. we consider peristaltic flow with no pressure gradient, given by putting $a_0 = 0$ and flow through a fixed tube with a prescribed pressure gradient, given by putting $\sigma = 0$.

6.1 *Peristaltic flow*

Flux through tube

In the case of channel flow the non-dimensional flux $l\bar{\psi}/\sigma$ has been calculated both to order $(\eta/h)^2$ and to order $(n/h)^4$ for a range of values of the two non-dimensional parameters $l\eta$ and lh. These results are displayed in figure 2 by showing the graphs of $l\bar{\psi}/\sigma$ against the ratio η/h for values of lh ranging from 0·25 to 2·0.

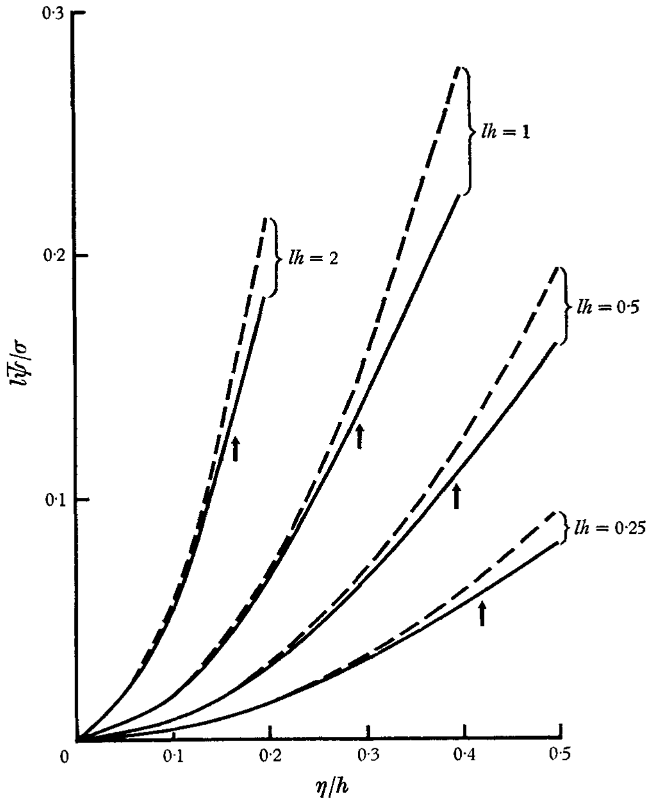

FIGURE 2. Peristalsis. Flux through channel. ---, $O(\eta/h)^2$; ——, $O(\eta/h)^4$.

For pipe flow, the non-dimensional flux $l^2\bar{\psi}/\sigma$ has been similarly calculated and the results are shown in the same way in figure 3.

Throughout the development of the theory there has been an implicit assumption that conditions ensuring the convergence of the various processes are satisfied. It is clear that, for a given value of lh, a perturbation solution in powers of η/h can be expected to converge only for values of η/h not exceeding some value depending on lh. Physically of course $\eta/h < 1$.

It can be seen from figures 2 and 3 that, for each value of lh, as η/h increases, the curves for the non-dimensional flux to order $(n/h)^2$ and to order $(\eta/h)^4$ begin to diverge appreciably which suggests that the limit to the convergence of the process has been reached. The calculations have also been carried out to order $(\eta/h)^6$ and the curve for $l\psi/\sigma$ to this order lies between the curves for $l\psi/\sigma$ to order $(\eta/h)^2$ and to order $(\eta/h)^4$. This suggests that the curve for order $(\eta/h)^2$ is the upper bound and the curve for order $(\eta/h)^4$ is the lower bound.

A point has been indicated for each value of lh, where the $(\eta/h)^4$ term first becomes one-tenth of the $(\eta/h)^2$ term. Hence if the application of the perturbation theory is limited to values of η/h below those indicated then the flux through the tube will be known to an accuracy of better than 10 %. This is an arbitrary criterion of course and Taylor (1951) uses 25% instead of 10 %.

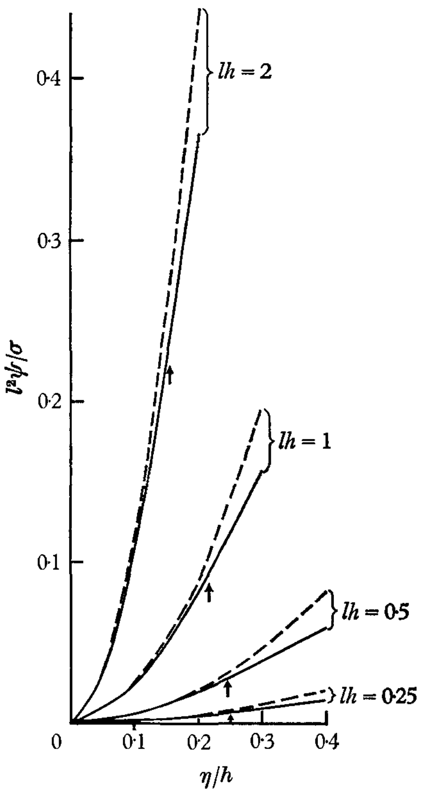

FIGURE 3. Peristalsis. Flux through pipe.
– – –, $O(\eta/h)^2$; ——, $O(\eta/h)^4$.

It will be seen from figures 2 and 3 that the indicated values of η/h decrease as lh increases.

Figure 2 shows that if η/h is constant, the non-dimensional flux per unit length normal to the plane of motion through a channel of mean breadth h, for a given wave velocity σ and given wavelength λ, is roughly proportional to h (i.e. to the area of the cross-section). If η is constant then the flux per unit length is roughly inversely proportional to h.

Figure 3 shows that if η/h is constant, the flux through a pipe of mean radius h is similarly roughly proportional to h^2 (i.e. to the area of cross-section) and that if η is constant then the flux is roughly independent of h.

Streamlines

Figure 4 shows the streamlines in two-dimensional flow for the case $lh = 0{\cdot}25$ and $\eta/h = 0{\cdot}1$ taking ψ to order $(\eta/h)^2$. The streamlines have only been drawn for positive y but of course they are symmetrical about the z-axis which has been taken as $\psi = 0$. The streamline $\psi = 0$ in addition to lying along the z-axis, also runs approximately perpendicular to the z-axis at $lz = 0{\cdot}56\pi$ and $lz = 1{\cdot}44\pi$ where $z = 0$ corresponds to a peak on the boundary.

At the boundary, $y = h + \eta \cos lz$, the streamlines are parallel to the y-axis.

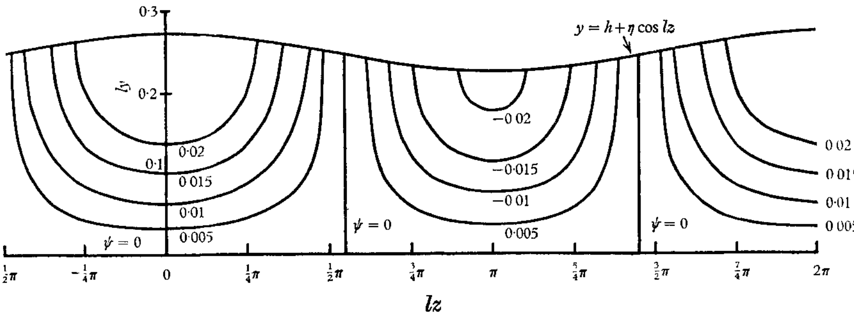

FIGURE 4. Peristalsis. Streamlines in channel flow with $lh = 0{\cdot}25$ and $l\eta = 0{\cdot}025$. Streamlines correspond to indicated values of $l\psi/\sigma$.

Velocity distributions

Figures 5 and 6 show the distribution of the velocity parallel to and perpendicular to the axis of the channel in two-dimensional flow for the case $lh = 0{\cdot}25$ and $\eta/h = 0{\cdot}1$. In both directions the flow is symmetrical about $lz = \pi$ (i.e. a trough in the boundary). The maximum velocity for the case considered is $0{\cdot}16\sigma$ along the axis of the channel and $0{\cdot}25\sigma$ at the boundary, perpendicular to the axis.

6.2. *Fixed boundary with prescribed pressure gradient* (*i.e.* $\sigma = 0$)

Flux through tube

The ratio $\overline{\psi}/\overline{\psi}_0$ has been computed for both channel flow and pipe flow for several values of lh and plotted against η/h in figures 7 and 8. In each case the flux is given to order $(\eta/h)^2$ and to order $(\eta/h)^4$.

As in the case of peristaltic flow the curves for $\overline{\psi}/\overline{\psi}_0$ to order $(\eta/h)^2$ and to order $(\eta/h)^4$ diverge appreciably. Points have been indicated on the curves where the $(\eta/h)^4$ term first becomes one-tenth of the $(\eta/h)^2$ term so that, for values of η/h below those indicated, the flux through the tube will be known to better than 10%.

It should be noted that increasing the amplitude η for a given h and l has opposite effects in the two cases considered. In peristaltic motion, the flux through the tube increases with η but with a prescribed pressure gradient and fixed boundary, the flux decreases as η is increased.

The flux of a viscous fluid through a uniform two-dimensional channel or through a uniform pipe of circular cross-section is given by $\overline{\psi}_0$ which is the value of $\overline{\psi}$ when $\eta = 0$.

For two-dimensional flow the flux per unit length normal to the plane of motion through a uniform channel of breadth $2h$ is $2\bar{\psi}_0$, while for axi-symmetrical flow the flux through a uniform pipe of radius h is $2\pi\bar{\psi}_0$. $\bar{\psi}_0$ is given in the two cases by (29) and (30) and the values given there can be expressed in terms of the original quantities by using (16), (17) and (13) noting, however, that P, the pressure drop over a wavelength should now be replaced by $P_0 2\pi/l$ where P_0 is the pressure drop per unit length. The resulting values of the flux are $2P_0h^3/3\mu$ for the channel and $\pi P_0 h^4/8\mu$ for the pipe.

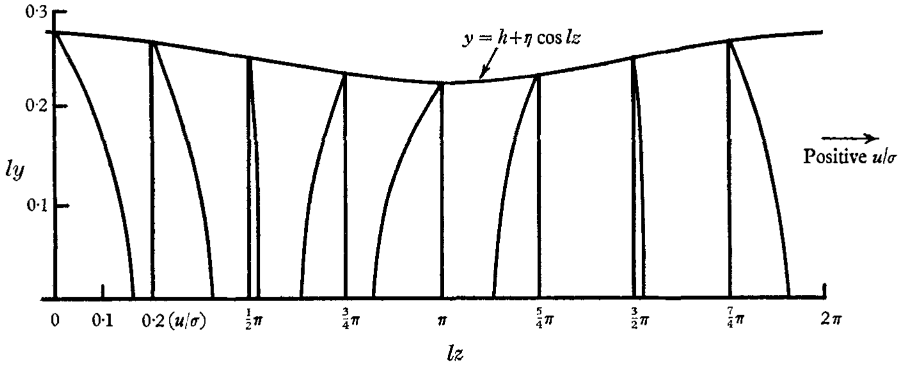

FIGURE 5. Peristalsis. Velocity distribution along the axis in channel flow with $lh = 0{\cdot}25$ and $l\eta = 0{\cdot}025$.

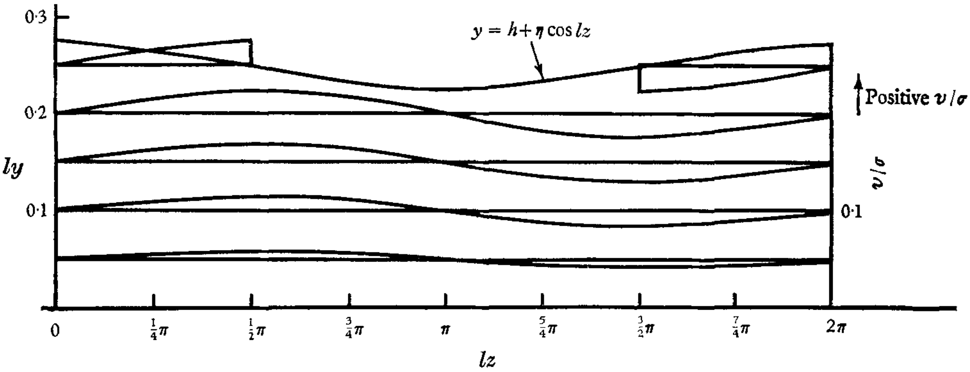

FIGURE 6. Peristalsis. Velocity distribution normal to the axis in channel flow with $lh = 0{\cdot}25$ and $l\eta = 0{\cdot}025$.

These are the well-known values obtained by solving the full Navier–Stokes equations of motion. For this simple type of flow, all the non-linear terms vanish and the Stokes equations give the flow exactly.

Streamlines

The streamlines in this case of steady flow follow the expected pattern, i.e. approximate cosine curves whose amplitude increases from 0 on the axis of symmetry to η on the boundary. Consequently they are not included in graphical form. Neither are the velocity distributions.

7. Other boundary conditions

The boundary conditions (5) used throughout the paper, require that the wall of the tube be extensible. The equations have also been solved for boundary conditions similar to those used by Taylor (1951). In the case of two-dimensional flow these correspond to an inextensible wall.

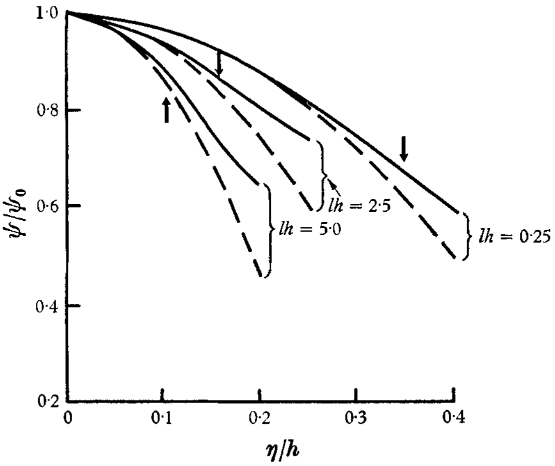

FIGURE 7. Fixed boundary. Flux through channel.
– – –, $O(\eta/h)^2$; ——, $O(\eta/h)^4$.

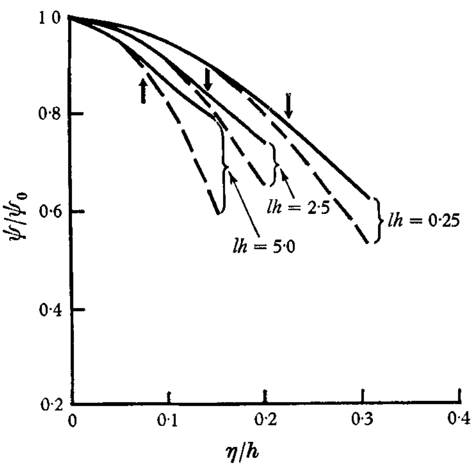

FIGURE 8. Fixed boundary. Flux through pipe.
– – –, $O(\eta/h)^2$; ——, $O(\eta/h)^4$.

These boundary conditions are, to order $(l\eta)^4$,

$$\frac{u}{\sigma} = -\frac{(l\eta)^4}{32} - \left(\frac{(l\eta)^2}{4} - \frac{(l\eta)^4}{8}\right)\cos 2lz - \frac{3(l\eta)^4}{64}\cos 4lz \tag{31}$$

and
$$\frac{v}{\sigma} = \left(l\eta - \frac{(l\eta)^3}{8}\right)\sin lz + \frac{(l\eta)^3}{8}\sin 3lz$$

on $y = h + \eta\cos lz$.

The difference between the two solutions is very small and can be ignored.

8. Conclusion

The perturbation method of solving equations (26) for the coefficients in the Fourier series (11) for the stream functions has been shown to be satisfactory and an indication of the range of convergence of the process has been obtained.

It has been shown further that a reasonable estimate of the flux through a tube (whether produced by peristaltic motion of the tube wall in the absence of a pressure gradient or by a constant pressure gradient in a fixed tube) is obtained from the first two terms of the series giving the flux in powers of the square of the relative variation in the radius. On the other hand, not unexpectedly, it has turned out that the process has only a moderate range of convergence. For applications to peristaltic motion this should not be an irksome restriction; we have been unable to find any data on the relative amplitudes in naturally occurring peristaltic motion but for the flow of blood along arteries the ratio of η/h is given by McDonald & Taylor (1959) as 0·04. Nevertheless, particularly for flow through fixed tubes, it would be desirable to have solutions for values of lh and η/h outside the range provided by the present method, and a more direct numerical approach to solving equations (26) is at present under consideration. Preliminary work has shown that for small η/h not many coefficients are needed and the two methods agree very closely. For large η/h more coefficients are needed and it becomes impractical to do all the calculations without the aid of a digital computer.

REFERENCES

BELINFANTE, D. C. 1962 *Proc. Camb. Phil. Soc.* **58**, 405.

BURNS, J. C. 1965 *Proceedings of the Second Australasian Conference on Hydraulics and Fluid Mechanics*, C. 57.

BURNS, J. C. 1966 *Am. Math. Mon.* **73** (4, Pt. II), 98.

GHEORGITA, ST. I. 1959 *Bull. math. Soc. Sci. math. phys. Répub pop. roum.* **3** (51), 283.

LANGLOIS, W. E. 1964 *Slow Viscous Flow.* New York: Macmillan.

MCDONALD, D. A. & TAYLOR, M. G. 1959 *Hydrodynamics of the Arterial Circulation. Progress in Biophysics and Biophysical Chemistry*, **9**, 107–73.

ROSENHEAD, L. 1963 (Editor) *Laminar Boundary Layers.* Oxford University Press.

TAYLOR, G. I. 1951 *Proc. Roy. Soc.* A **209**, 447.

WEINSTEIN, A. 1953 *Bull. Am. Math. Soc.* **59**, 20.